Strong chains of subsets of ω_1 of length ω_3

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Given an ordinal δ, a *strong chain of functions from* $\omega_1^{\omega_1}$ *of length* δ is a sequence $(f_\alpha : \alpha < \delta)$ of functions $f_\alpha : \omega_1 \to \omega_1$ such that for all $\alpha < \beta < \delta$, $\{\nu \in \omega_1 : f_\alpha(\nu) \geq f_\beta(\nu)\}$ is finite. Hajnal in the 1990s asked if the existence of a strong chain of functions from $\omega_1^{\omega_1}$ of length ω_2 was consistent. The question was answered affirmatively by Koszmider [1], and later, Veličković and Venturi [2] simplified the proof by using Neeman's two-type side condition method. The question of finding longer strong chains remained open and was seen as a test question for finding side conditions of models of three types.

Given an ordinal δ, a *strong chain of subsets of* ω_1 *of length* δ is a sequence $(X_\alpha : \alpha < \delta)$ of subsets of ω_1 such that $X_\alpha \setminus X_\beta$ is finite and $|X_\beta \setminus X_\alpha| = \aleph_1$, for all $\alpha < \beta < \delta$. The existence of strong chains of subsets of ω_1 follows from the existence of strong chains of functions from $\omega_1^{\omega_1}$, by identifying each subset with its characteristic function. So, an easier question would be to ask whether it is consistent to have strong chains of subsets of ω_1 of length $> \omega_2$. In this talk, I will present a joint work with David Asperó in which we answer this question affirmatively, but the same ideas should lead to the consistency of the existence of a strong chain of functions from $\omega_1^{\omega_1}$ of length ω_3. In particular, we define a forcing with symmetric systems of models of two types as side conditions, which preserves all cardinals and forces a strong chain of subsets of ω_1 of length ω_3.

References

[1] Piotr Koszmider. On strong chains of uncountable functions. *Israel Journal of Mathematics*, 118(1):289–315, 2000.

[2] Boban Veličković and Giorgio Venturi. Proper forcing remastered. *arXiv preprint arXiv:1110.0610*, 2011.